

PARANORMAL WEIGHTED CONDITIONAL TYPE OPERATORS

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ABSTRACT. In this paper, some sub-classes of paranormal weighted conditional expectation type operators, such as $*$ -paranormal, quasi- $*$ -paranormal and (n, k) -quasi- $*$ -paranormal weighted conditional expectation type operators on $L^2(\Sigma)$ are investigated. Also, some applications about the spectrum, point spectrum, joint point spectrum, approximate point spectrum and joint approximate point spectrum of these classes are presented.

1. Introduction and Preliminaries

Theory of weighted conditional expectation type operators is one of important arguments in the connection of operator theory and measure theory. Weighted conditional expectations have been studied in an operator theoretic setting, by many authors, for example, De pagter and Grobler [7] and Rao [13, 14], as positive operators acting on L^p -spaces or Banach function spaces. In [11], S.-T. C. Moy characterized all operators on L^p of the form $f \rightarrow E(fg)$ for g in L^q with $E(|g|)$ bounded. Also, some results about these operators can be found in [1, 8, 9]. In [2] P.G. Dodds, C.B. Huijsmans and B. de Pagter showed that lots of operators are of the form of weighted conditional type operators. In [7] a class of operators which factorizes through weighted conditional type operators is investigated. This class of operators includes operators such as kernel operators and order continuous Riesz homomorphisms. Also, we investigated some classical properties of these operators on L^p -spaces in [3, 4, 5].

Let (X, Σ, μ) be a σ -finite measure space. For a sub- σ -finite algebra $\mathcal{A} \subseteq \Sigma$, the conditional expectation operator associated with \mathcal{A} is the mapping $f \rightarrow E^{\mathcal{A}}f$, defined for all non-negative measurable functions f as well as for all $f \in L^2(\Sigma)$, where $E^{\mathcal{A}}f$, by the Radon-Nikodym theorem, is the unique \mathcal{A} -measurable function satisfying $\int_A f d\mu = \int_A E^{\mathcal{A}}f d\mu$, $\forall A \in \mathcal{A}$. As an operator on $L^2(\Sigma)$, $E^{\mathcal{A}}$ is idempotent and $E^{\mathcal{A}}(L^2(\Sigma)) = L^2(\mathcal{A})$. This operator will play a major role in our work. Let $f \in L^0(\Sigma)$, then f is said to be conditionable with respect to E if $f \in \mathcal{D}(E) := \{g \in L^0(\Sigma) : E(|g|) \in L^0(\mathcal{A})\}$ in which $L^0(\Sigma)$ is the vector space of all equivalence classes of almost everywhere finite valued measurable functions on X . Throughout this paper we take u and w in $\mathcal{D}(E)$. If there is no possibility of confusion, we write $E(f)$ in place of $E^{\mathcal{A}}(f)$. A detailed discussion about this operator may be found in [12]. All comparisons between two functions or two sets

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are to be interpreted as holding up to a μ -null set. The support of a measurable function f is defined as $S(f) = \{x \in X; f(x) \neq 0\}$.

Let \mathcal{H} be the infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded operators on \mathcal{H} . An operator $T \in \mathcal{L}(\mathcal{H})$ is a partial isometry if $\|Th\| = \|h\|$ for h orthogonal to the kernel of T . It is known that an operator T on a Hilbert space is partial isometry if and only if $TT^*T = T$. Every operator T on a Hilbert space \mathcal{H} can be decomposed into $T = U|T|$ with a partial isometry U , where $|T| = (T^*T)^{\frac{1}{2}}$. U is determined uniquely by the kernel condition $\mathcal{N}(U) = \mathcal{N}(|T|)$. Then this decomposition is called the polar decomposition. The Aluthge transformation \hat{T} of the operator T is defined by $\hat{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. The operator T is said to be a positive operator and written as $T \geq 0$, if $\langle Th, h \rangle \geq 0$, for all $h \in \mathcal{H}$.

In this paper we will be concerned with characterizing weighted conditional expectation type operators on $L^2(\Sigma)$ in terms of membership of the partial paranormal classes. Also, we prove that the point spectrum and joint point spectrum of the weighted conditional type operators M_wEM_u are the same, when u, w satisfy a mild condition. Here is a brief review of what constitutes membership for an operator T on a Hilbert space in each classes:

(i) T is called paranormal if for all unit vectors x in H , $\|Tx\|^2 \leq \|T^2x\|$ or equivalently

$$T^{*2}T^2 - 2\lambda T^*T + \lambda^2 I \geq 0,$$

for all $\lambda > 0$;

(ii) T is called M -paranormal, if there exists $M > 0$ such that for all unit vectors x in H , $\|Tx\|^2 \leq M\|T^2x\|$ or equivalently $M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2 I \geq 0$, for all $\lambda > 0$.

(iii) T is called $*$ -paranormal, if $\|T^*x\|^2 \leq \|T^2x\|$ for all unit vector $x \in H$ or equivalently

$$T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I \geq 0,$$

for all $\lambda > 0$;

(iv) T is called quasi- $*$ -paranormal, if it satisfies the following inequality:

$$\|T^*Tx\|^2 \leq \|T^3x\| \cdot \|Tx\|$$

for all $x \in H$ or equivalently

$$T^*(T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I)T \geq 0,$$

for all $\lambda > 0$;

(v) T is called (n, k) -quasi- $*$ -paranormal if

$$\|T^{1+n}(T^kx)\|^{\frac{1}{1+n}} \|T^kx\|^{\frac{n}{n+1}} \geq \|T^*T^kx\|$$

for all $x \in H$, or equivalently

$$T^{*k}T^{*(1+n)}T^{1+n}T^k - (1+n)\mu^n T^{*k}TT^*T^k + n\mu^{n+1}T^{*k}T^k \geq 0$$

for all $\mu > 0$;

(vi) T is called absolute- k -paranormal for each $k > 0$ if

$$\| |T|^k T x \| \geq \| T x \|^{k+1}$$

for every unit vector $x \in H$ or equivalently

$$T^* |T|^{2k} T - (k+1) \lambda^k |T|^2 + k \lambda^{k+1} I \geq 0$$

for all $\lambda > 0$;

2. Partial paranormal weighted conditional type operators

First we recall some results of [2] that state our results is valid for a large class of linear operators. Let (X, Σ, μ) be a finite measure space, then $L^\infty(\Sigma) \subseteq L^2(\Sigma) \subseteq L^1(\Sigma)$ and $L^2(\Sigma)$ is an order ideal of measurable functions on (X, Σ, μ) . Thus by propositions (3.1, 3.3, 3.6) of [2] we have theorems A, B, C:

Theorem A. If T is a linear operator on $L^2(\Sigma)$ for which

- (i) $Tf \in L^\infty(\Sigma)$ whenever $f \in L^\infty(\Sigma)$.
- (ii) $\|Tf_n\|_1 \rightarrow 0$ for all sequences $\{f_n\}_{n=1}^\infty \subseteq L^2(\Sigma)$ such that $|f_n| \leq g$ ($n = 1, 2, 3, \dots$) for some $g \in L^2(\Sigma)$ and $f_n \rightarrow 0$ a.e.,
- (iii) $T(f.Tg) = Tf.Tg$ for all $f \in L^\infty(\Sigma)$ and all $g \in L^2(\Sigma)$,

then there exists a σ -subalgebra \mathcal{A} of Σ and there exists $w \in L^2(\Sigma)$ such that $Tf = E^{\mathcal{A}}(wf)$ for all $f \in L^2(\Sigma)$.

Theorem B. For a linear operator $T : L^2(\Sigma) \rightarrow L^2(\Sigma)$ the following statement are equivalent.

- (i) T is positive and order continuous, $T^2 = T$, $T1 = 1$ and the range $\mathcal{R}(T)$ of T is a sublattice.
- (ii) There exist a σ -subalgebra \mathcal{A} of Σ and a function $0 \leq w \in L^2(\Sigma)$ with $E^{\mathcal{A}}(w) = 1$ such that $Tf = E^{\mathcal{A}}(wf)$ for all $f \in L^2(\Sigma)$.

Theorem C. For a linear operator $T : L^2(\Sigma) \rightarrow L^2(\Sigma)$ the following statement are equivalent.

- (i) T is a positive and order continuous projection onto a sublattice such that $T1$ is strictly positive.
- (ii) There exist a σ -subalgebra \mathcal{A} of Σ , $0 \leq w \in L^2(\Sigma)$ and a strictly positive function $k \in L^1(\Sigma)$ with $E^{\mathcal{A}}(wk) = 1$ such that $Tf = E^{\mathcal{A}}(wf)$ for all $f \in L^2(\Sigma)$.

Moreover, if we choose k such that $E^{\mathcal{A}}(k) = 1$, then both w and k are uniquely determined by T .

Here, we recall some properties of weighted conditional type operators, that we have proved in [5].

The operator $T = M_w E M_u$ is bounded on $L^2(\Sigma)$ if and only if $(E|w|^2)^{\frac{1}{2}}(E|u|^2)^{\frac{1}{2}} \in L^\infty(\mathcal{A})$, and in this case its norm is given by $\|T\| = \|(E(|w|^2))^{\frac{1}{2}}(E(|u|^2))^{\frac{1}{2}}\|_\infty$. The unique polar decomposition of bounded operator $T = M_w E M_u$ is $U|T|$, where

$$|T|(f) = \left(\frac{E(|w|^2)}{E(|u|^2)} \right)^{\frac{1}{2}} \chi_S \bar{u} E(uf)$$

and

$$U(f) = \left(\frac{\chi_{S \cap G}}{E(|w|^2)E(|u|^2)} \right)^{\frac{1}{2}} w E(uf),$$

for all $f \in L^2(\Sigma)$, where $S = S(E(|u|^2))$ and $G = S(E(|w|^2))$.

In the sequel some necessary and sufficient conditions for weighted conditional type operator $M_w E M_u$ to be M -paranormal, quasi- $*$ -paranormal, absolute- k -paranormal and (n, k) -quasi- $*$ -paranormal will be presented.

Theorem 2.1. Let $T = M_w E M_u$ be a bounded operator on $L^2(\Sigma)$. Then

(i) If T is M -paranormal, we have

$$(M^2 |E(uw)|^2 E(|w|^2) - 2\lambda E(|w|^2)) |E(u)|^2 + \lambda^2 \geq 0,$$

for all $\lambda > 0$.

(ii) If $M^2 |E(uw)|^2 E(|w|^2) - 2\lambda E(|w|^2) \geq 0$ for all $\lambda > 0$, then T is M -paranormal.

Proof. (i) By induction and by Lemma we have

$$\begin{aligned} T^* T &= M_{E(|w|^2)} M_{\bar{u}} E M_u, \\ T^{*^2} T^2 &= M_{|E(uw)|^2 E(|w|^2)} M_{\bar{u}} E M_u. \end{aligned}$$

So for every $\lambda > 0$ and $M > 0$

$$\begin{aligned} M^2 T^{*^2} T^2 - 2\lambda T^* T + \lambda^2 I &= M^2 M_{|E(uw)|^2 E(|w|^2)} M_{\bar{u}} E M_u - 2\lambda M_{E(|w|^2)} M_{\bar{u}} E M_u + \lambda^2 I \\ &= (M_{M^2 |E(uw)|^2 E(|w|^2) - 2\lambda E(|w|^2)}) M_{\bar{u}} E M_u + \lambda^2 I. \end{aligned}$$

Let $\alpha = M^2 |E(uw)|^2 E(|w|^2) - 2\lambda E(|w|^2)$. Then for every $f \in L^2(\Sigma)$ we get

$$\langle M_\alpha M_{\bar{u}} E M_u f + \lambda^2 f, f \rangle = \int_X \alpha |E(uf)|^2 d\mu + \int_X \lambda^2 |f|^2 d\mu$$

This implies that if $\alpha \geq 0$ a.e, then T is M -paranormal.

(ii) If T is M -paranormal, then for all $f \in L^2(\mathcal{A})$

$$\begin{aligned}
\langle M_\alpha M_{\bar{u}} E M_u f + \lambda^2 f, f \rangle &= \int_X \alpha |E(uf)|^2 d\mu + \int_X \lambda^2 |f|^2 d\mu \\
&= \int_X \alpha |E(u)|^2 |f|^2 d\mu + \int_X \lambda^2 |f|^2 d\mu \\
&= \int_X (\alpha |E(u)|^2 + \lambda^2) |f|^2 d\mu \geq 0.
\end{aligned}$$

Therefore $\alpha |E(u)|^2 + \lambda^2 \geq 0$ a.e.,

Corollary 2.2. Let $T = M_w E M_u$ be a bounded operator on $L^2(\Sigma)$. Then

(i) If T is paranormal, we have

$$(|E(uw)|^2 E(|w|^2) - 2kE(|w|^2)) |E(u)|^2 + k^2 \geq 0,$$

(ii) If $|E(uw)|^2 E(|w|^2) - 2kE(|w|^2) \geq 0$, then T is paranormal.

Corollary 2.3. Let $u \in L^0(\mathcal{A})$ and $T = M_w E M_u$ be a bounded operator on $L^2(\Sigma)$. Then:

(i) T is M -paranormal if and only if

$$(M^2 |uE(w)|^2 E(|w|^2) - 2\lambda E(|w|^2)) |u|^2 + \lambda^2 \geq 0,$$

for all $\lambda > 0$.

(ii) T is paranormal if and only if

$$(|uE(w)|^2 E(|w|^2) - 2\lambda E(|w|^2)) |u|^2 + \lambda^2 \geq 0,$$

for all $\lambda > 0$.

Proof. Since $|E(F)|^2 \leq E(|f|^2)$ for every $f \in L^2(\Sigma)$, then by similar method of Theorem 2.2 we get the proof.

The definition of quasi- $*$ -paranormal and $*$ -paranormal operators shows that, if T is quasi- $*$ -paranormal, then $T|_{\overline{\mathcal{R}(T)}}$ is $*$ -paranormal. Therefore, if T has dense range, then T is quasi- $*$ -paranormal if and only if is $*$ -paranormal. In the next theorem we give a necessary and sufficient condition for $M_w E M_u$ to be quasi- $*$ -paranormal.

Theorem 2.4. Let $T = M_w E M_u$ be a bounded operator on $L^2(\Sigma)$. Then T is quasi- $*$ -paranormal if and only if

$$E(|u|^2) E(|w|^2) \leq |E(uw)|^2 \text{ a.e., on } G.$$

Where $G = S(E(|w|^2))$.

Proof. By direct computations we have

$$T^n f = (E(uw))^{n-1} w E(u f), \quad T^{*n} f = (\overline{E(uw)})^{n-1} \bar{u} E(\bar{w} f),$$

for all $f \in L^2(\Sigma)$ and $n \in \mathbb{N}$. So we get that

$$(T^* T)^2 = M_{\bar{u} E(|u|^2) \chi_S(E(|w|^2))^2} E M_u,$$

$$T^* T = M_{\bar{u} E(|w|^2)} E M_u,$$

$$T^{*3} T^3 = M_{|E(uw)|^4 E(|w|^2)} M_{\bar{u}} E M_u.$$

Therefore T is quasi- $*$ -paranormal if and only if

$$\begin{aligned} & M_{|E(uw)|^4 E(|w|^2)} M_{\bar{u}} E M_u - 2\lambda M_{\bar{u} E(|u|^2) \chi_S(E(|w|^2))^2} E M_u + \lambda^2 M_{\bar{u} E(|w|^2)} E M_u \\ &= (M_{|E(uw)|^4 E(|w|^2)} - 2\lambda M_{E(|u|^2) \chi_S(E(|w|^2))^2} + \lambda^2 M_{E(|w|^2)}) M_{\bar{u}} E M_u \geq 0. \end{aligned}$$

This implies that T is quasi- $*$ -paranormal if and only if for all $f \in L^2(\Sigma)$ and $\lambda > 0$

$$\begin{aligned} 0 &\leq \langle M_{|E(uw)|^4 E(|w|^2)} f - 2\lambda M_{E(|u|^2) \chi_S(E(|w|^2))^2} f + \lambda^2 M_{E(|w|^2)} f, f \rangle \\ &= \langle M_{(|E(uw)|^4 E(|w|^2) - 2\lambda E(|u|^2) \chi_S(E(|w|^2))^2 + \lambda^2 E(|w|^2))} f, f \rangle, \\ &\iff M_{(|E(uw)|^4 E(|w|^2) - 2\lambda E(|u|^2) \chi_S(E(|w|^2))^2 + \lambda^2 E(|w|^2))} \geq 0 \\ &\iff |E(uw)|^4 E(|w|^2) - 2\lambda E(|u|^2) \chi_S(E(|w|^2))^2 + \lambda^2 E(|w|^2) \geq 0 \\ &\iff (E(|u|^2))^2 (E(|w|^2))^4 - (E(|w|^2))^2 |E(uw)|^4 \geq 0 \\ &\iff E(|u|^2) E(|w|^2) \leq |E(uw)|^2 \text{ on } G, \end{aligned}$$

where we have used the fact that $T_1 T_2 \geq 0$ if $T_1 \geq 0$, $T_2 \geq 0$ and $T_1 T_2 = T_2 T_1$ for all $T_i \in \mathcal{B}(\mathcal{H})$, positivity of $M_{\bar{u}} E M_u$ and $M_{\alpha} M_{\bar{u}} E M_u = M_{\bar{u}} E M_u M_{\alpha}$ such that

$$\alpha = |E(uw)|^4 E(|w|^2) - 2\lambda E(|u|^2) \chi_S(E(|w|^2))^2 + \lambda^2 E(|w|^2).$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is a quasi- $*$ - A -class operator if $T^* |T|^2 |T| \geq T^* |T^*|^2 T$. And T is an A -class operator if $|T|^2 \leq |T^*|^2$. One can see [6] for more details. In [4] we studied quasi- $*$ - A -class weighted conditional type operators. Here we get that quasi- $*$ - A -class and quasi- $*$ -paranormal weighted conditional type operators are coincided.

Theorem 2.5. Let $T = M_w E M_u$ be a bounded operator on $L^2(\Sigma)$ and $G = X$. Then the followings are mutually equivalent:

- (a) T is quasi- $*$ -paranormal;
- (b) T is a quasi- $*$ - A -class operator;
- (c) $E(|u|^2) E(|w|^2) \leq |E(uw)|^2$ a.e.,

Moreover, if $S(E(u)) = X = G$, then (a), (b), (c) and (d) are mutually equivalent, where

(d) T is an A -class operator.

Proof. This is a direct consequence of Theorem 2.4 and Theorems 2.6 and 2.8 of [4].

Theorem 2.6. Let $T = M_w E M_u$ be a bounded operator on $L^2(\Sigma)$. Then,

(i) If T is absolute- k -paranormal, we have

$$(|E(uw)|^2 (E(|u|^2))^{k-1} \cdot \chi_S(E(|w|^2))^k |E(u)|^2 - (k+1)\lambda^k E(|w|^2) |E(u)|^2 + k\lambda^{k+1} \geq 0,$$

for all $\lambda > 0$.

(ii) If $|E(uw)|^2 (E(|u|^2))^{k-1} \cdot \chi_S(E(|w|^2))^k - (k+1)\lambda^k E(|w|^2) |E(u)|^2 \geq 0$ for all $\lambda > 0$, then T is absolute- k -paranormal.

Proof. (i) By similar methods of last theorems we have

$$\begin{aligned} T^* |T|^{2k} T &= M_{(E(|u|^2))^{k-1} \chi_S(E(|w|^2))^k |E(uw)|^2} M_{\bar{u}} E M_u, \\ T^* T &= M_{E(|w|^2)} M_{\bar{u}} E M_u, \end{aligned}$$

If T is absolute- k -paranormal, then for all $f \in L^2(\mathcal{A})$

$$\begin{aligned} 0 &\leq \langle M_{(E(|u|^2))^{k-1} \chi_S(E(|w|^2))^k |E(uw)|^2} M_{\bar{u}} E M_u f - (k+1)\lambda^k M_{E(|w|^2)} M_{\bar{u}} E M_u f + k\lambda^{k+1} f, f \rangle \\ &= \int_X ((E(|u|^2))^{k-1} \chi_S(E(|w|^2))^k |E(uw)|^2 - (k+1)\lambda^k E(|w|^2)) \bar{u} E(u) f \bar{f} d\mu + k\lambda^{k+1} \int_X |f|^2 d\mu \\ &= \int_X ((E(|u|^2))^{k-1} \chi_S(E(|w|^2))^k |E(uw)|^2 |E(u)|^2 - (k+1)\lambda^k E(|w|^2) |E(u)|^2 + k\lambda^{k+1}) |f|^2 d\mu. \end{aligned}$$

So we get that

$$(E(|u|^2))^{k-1} \chi_S(E(|w|^2))^k |E(uw)|^2 |E(u)|^2 - (k+1)\lambda^k E(|w|^2) |E(u)|^2 + k\lambda^{k+1} \geq 0.$$

(ii) The operator T is absolute- k -paranormal if for all $f \in L^2(\Sigma)$

$$\begin{aligned} 0 &\leq T^* |T|^{2k} T - (k+1)\lambda^k |T|^2 + k\lambda^{k+1} I \geq 0 \\ &= \langle M_{(E(|u|^2))^{k-1} \chi_S(E(|w|^2))^k |E(uw)|^2} M_{\bar{u}} E M_u f - (k+1)\lambda^k M_{E(|w|^2)} M_{\bar{u}} E M_u f + k\lambda^{k+1} f, f \rangle \\ &= \int_X ((E(|u|^2))^{k-1} \chi_S(E(|w|^2))^k |E(uw)|^2 - (k+1)\lambda^k E(|w|^2)) |E(u f)|^2 d\mu + k\lambda^{k+1} \int_X |f|^2 d\mu. \end{aligned}$$

This implies that if $|E(uw)|^2 (E(|u|^2))^{k-1} \cdot \chi_S(E(|w|^2))^k - (k+1)\lambda^k E(|w|^2) |E(u)|^2 \geq 0$, then T is absolute- k -paranormal.

Corollary 2.7. Let $u \in L^0(\mathcal{A})$ and $T = M_w E M_u$ be a bounded operator on $L^2(\Sigma)$. Then T is absolute- k -paranormal if and only if

$$(|u E(w)|^2 |u|^{2k-2} \cdot \chi_S(E(|w|^2))^k |u|^2 - (k+1)\lambda^k E(|w|^2) |u|^2 + k\lambda^{k+1} \geq 0,$$

for all $\lambda > 0$.

Proof. Since $|E(F)|^2 \leq E(|f|^2)$ for every $f \in L^2(\Sigma)$, then by similar method of Theorem 2.6 we get the proof.

Theorem 2.8. Let $T = M_w EM_u$ be a bounded operator on $L^2(\Sigma)$. Then T is (n, k) -quasi- $*$ -paranormal if and only if $\alpha_\mu \geq 0$ for all $\mu > 0$, where

$$\begin{aligned} \alpha_k &= |E(uw)|^{2(n+k)} E(|w|^2) - (1+n)\mu^n |E(uw)|^{2(k-1)} \chi_{S_0} (E(|w|^2))^2 E(|u|^2) \\ &\quad + \mu^{1+n} |E(uw)|^{2(k-1)} \chi_{S_0} E(|w|^2), \end{aligned}$$

and $S_0 = S(E(uw))$.

Proof. By similar methods of last theorems we have

$$\begin{aligned} T^{*k} T^{*(1+n)} T^{1+n} T^k &= M_{|E(uw)|^{2(n+k)} E(|w|^2)} M_{\bar{u}} EM_u, \\ T^{*k} T T^* T^k &= M_{|E(uw)|^{2(k-1)} (E(|w|^2))^2 E(|u|^2)} M_{\bar{u}} EM_u, \\ T^{*k} T^k &= M_{|E(uw)|^{2(k-1)} E(|w|^2)} M_{\bar{u}} EM_u. \end{aligned}$$

This implies that T is (n, k) -quasi- $*$ -paranormal if and only if

$$\begin{aligned} (M_{|E(uw)|^{2(n+k)} E(|w|^2)} - (1+n)\mu^n M_{|E(uw)|^{2(k-1)} (E(|w|^2))^2 E(|u|^2)} \\ + n\mu^{n+1} M_{|E(uw)|^{2(k-1)} E(|w|^2)}) M_{\bar{u}} EM_u \geq 0. \end{aligned}$$

This inequality holds if and only if

$$\begin{aligned} M_{|E(uw)|^{2(n+k)} E(|w|^2)} - (1+n)\mu^n M_{|E(uw)|^{2(k-1)} (E(|w|^2))^2 E(|u|^2)} \\ + n\mu^{n+1} M_{|E(uw)|^{2(k-1)} E(|w|^2)} \geq 0. \end{aligned}$$

where we have used the fact that $T_1 T_2 \geq 0$ if $T_1 \geq 0$, $T_2 \geq 0$ and $T_1 T_2 = T_2 T_1$ for all $T_i \in \mathcal{B}(\mathcal{H})$, positivity of $M_{\bar{u}} EM_u$ and $M_{\alpha_\mu} M_{\bar{u}} EM_u = M_{\bar{u}} EM_u M_{\alpha_\mu}$ such that

$$\begin{aligned} \alpha_\mu &= |E(uw)|^{2(n+k)} E(|w|^2) - (1+n)\mu^n |E(uw)|^{2(k-1)} \chi_{S_0} (E(|w|^2))^2 E(|u|^2) \\ &\quad + n\mu^{n+1} |E(uw)|^{2(k-1)} \chi_{S_0} E(|w|^2). \end{aligned}$$

Therefore the operator T is (n, k) -quasi- $*$ -paranormal if and only if $M_{\alpha_\mu} \geq 0$ if and only if $\alpha_\mu \geq 0$ for all $\mu > 0$.

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be n - $*$ -paranormal if $\|T^{n+1}x\|^{\frac{1}{n+1}} \|x\|^{\frac{n}{n+1}} \geq \|T^*x\|$ for all $x \in \mathcal{H}$. Also, T is called k -quasi- $*$ -paranormal if $\|T^2(T^k x)\|^{\frac{1}{2}} \|T^k x\|^{\frac{1}{2}} \geq \|T^*(Tx)\|$ for all $x \in \mathcal{H}$. So, we get the following corollaries.

Corollary 2.9. Let $T = M_w EM_u$ be a bounded operator on $L^2(\Sigma)$. Then T is n - $*$ -paranormal if and only if $\alpha_\mu \geq 0$ for all $\mu > 0$, where

$$\begin{aligned}\alpha_\mu &= |E(uw)|^{2(n)} E(|w|^2) - (1+n)\mu^n |E(uw)|^{-2} \chi_{S_0}(E(|w|^2))^2 E(|u|^2) \\ &\quad + n\mu^{n+1} |E(uw)|^{-2} \chi_{S_0} E(|w|^2).\end{aligned}$$

Corollary 2.10. Let $T = M_w E M_u$ be a bounded operator on $L^2(\Sigma)$. Then T is k -quasi- $*$ -paranormal if and only if $\alpha_\mu \geq 0$ for all $\mu > 0$, where

$$\begin{aligned}\alpha_\mu &= |E(uw)|^{2(1+k)} E(|w|^2) - 2\mu^n |E(uw)|^{2(k-1)} \chi_{S_0}(E(|w|^2))^2 E(|u|^2) \\ &\quad + \mu^2 |E(uw)|^{2(k-1)} \chi_{S_0} E(|w|^2).\end{aligned}$$

3. Some Applications

For $T \in \mathcal{L}(\mathcal{H})$, let $\sigma_p(T)$, $\sigma_{jp}(T)$, $\sigma_a(T)$ and $\sigma_{ja}(T)$ denote the point spectrum, joint point spectrum, approximate point spectrum and joint approximate point spectrum of T . T is called isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T . Let $\mu \in \mathbb{C}$ be an isolated point of $\sigma(T)$. Then the riesz idempotent E_μ of T with respect to μ is defined by

$$E_\mu := \frac{1}{2\pi i} \int_{\partial D_\mu} (\mu I - T)^{-1} d\mu,$$

where D_μ is the closed disk centered at μ which contains no other points of $\sigma(T)$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP at λ_0 for brevity) if for every open neighborhood U of λ_0 , the only analytic function $f : U \rightarrow H$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the constant function $f \equiv 0$. Here we recall some spectral results about $M_w E M_u$. Then we get some conclusions for quasi- $*$ -paranormal and

Theorem 3.1. [4] Let $T = M_w E M_u : L^2(\Sigma) \rightarrow L^2(\Sigma)$. Then

(a)

$$\sigma(M_w E M_u) \setminus \{0\} = \text{ess range}(E(uw)) \setminus \{0\}.$$

(b) If $S \cap G = X$, then

$$\sigma(M_w E M_u) = \text{ess range}(E(uw)),$$

where $S = S(E(|u|^2))$ and $G = S(E(|w|^2))$.

(c)

$$\sigma_p(M_w E M_u) \setminus \{0\} = \{\lambda \in \mathbb{C} \setminus \{0\} : \mu(A_{\lambda,w}) > 0\},$$

where $A_{\lambda,w} = \{x \in X : E(uw)(x) = \lambda\}$.

Proposition 3.2, If $|E(uw)|^2 \geq E(|u|^2)E(|w|^2)$, then

$$\sigma_p(M_w E M_u) = \sigma_{jp}(M_w E M_u).$$

Proof, This is a direct consequence of theorems 2.8 and 3.4 of [4].

Corollary 3.3. If M_wEM_u is quasi- $*$ -paranormal and $G = X$ or is an A -class operator and $S(E(u)) = X$ or is a quasi- $*$ - A -class operator, then $\sigma_p(M_wEM_u) = \sigma_{jp}(M_wEM_u)$.

Proof, This is a direct consequence of Proposition 3.1, Theorems 2.3 and 2.4.

For all $f \in L^2(\Sigma)$ we have

$$\begin{aligned} \langle wE(uf), f \rangle &= \int_X wE(uf) \bar{f} d\mu \\ &= \int_X ufE(w\bar{f}) d\mu \\ &= \int_X f\bar{u}E(\bar{w}f) d\mu \\ &= \langle f, \bar{u}E(\bar{w}f) \rangle. \end{aligned}$$

This implies that $(M_wEM_u)^* = M_{\bar{u}}EM_{\bar{w}}$.

Proposition 3.4. Let $|E(uw)|^2 \geq E(|u|^2)E(|w|^2)$ on G . Then

(a) Every non-zero isolated point of $\text{ess range}(E(uw))$ is a simple pole of the resolvent of M_wEM_u .

(b) If μ is a non-zero isolated point of $\text{ess range}(E(uw))$ and E_μ is the Riesz idempotent of M_wEM_u with respect to μ . Then E_μ is self-adjoint if and only if $N(M_wEM_u - \mu) \subseteq N(M_{\bar{u}}EM_{\bar{w}} - \bar{\mu})$.

Proof, By using the results of [10] and Theorem 2.3 we get the proof.

The next corollary is a direct consequence of Theorem 2.6 and the results of [16].

Corollary 3.5. Let M_wEM_u be a bounded operator on $L^2(\Sigma)$. If $\alpha_\mu \geq 0$ for all $\mu > 0$, where

$$\begin{aligned} \alpha_k &= |E(uw)|^{2(n+k)}E(|w|^2) - (1+n)\mu^n|E(uw)|^{2(k-1)}\chi_{S_0}(E(|w|^2))^2E(|u|^2) \\ &\quad + \mu^{1+n}|E(uw)|^{2(k-1)}\chi_{S_0}E(|w|^2), \end{aligned}$$

and $S_0 = S(E(uw))$,
then

(a) For every $0 \neq \lambda \in \mathbb{C}$ we have

$$\ker(M_w EM_u - \lambda) \subseteq \ker(M_{\bar{u}} EM_{\bar{w}} - \bar{\lambda}).$$

(b)

$$\sigma_{jp}(M_w EM_u) \setminus \{0\} = \sigma_p(M_w EM_u) \setminus \{0\},$$

and

$$\sigma_{ja}(M_w EM_u) \setminus \{0\} = \sigma_a(M_w EM_u) \setminus \{0\}.$$

(c) If $\lambda \neq \mu$, then

$$\ker(M_w EM_u - \lambda) \perp \ker(M_w EM_u - \mu).$$

(d) For every $0 \neq \lambda \in \mathbb{C}$ we have

$$\ker(M_w EM_u - \lambda) = \ker(M_w EM_u - \lambda)^2,$$

and

$$\ker((M_w EM_u)^{k+1}) = \ker((M_w EM_u)^{k+2}).$$

(e) The operator $M_w EM_u$ has SVEP.

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